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On the Initial-Boundary Value Problem of Some Geophysical Fluid Flows

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The so called "energy method" of analyzing the properties of difference equations is systematically applied to certain conventional representations of the prototype advection, vorticity and shallow water equations of geophysical flows. Conditions governing the boundedness of the solutions to the difference equations are derived and these are used to obtain suitable consistent formulations for the boundary conditions on open (purely geometric) fixed surfaces immersed in the fluid. Experiments conducted with the derived boundary conditions indicate that they do not engender boundary instabilities. In particular, successful results were obtained with the shallow water equations for both quasinondivergent and almost irrotational initial flow conditions. The results suggest that the energy method constitutes a viable and useful approach to the "open" boundary problem posed in some geophysical fluid flow studies.

I. INTRODUCTION

It is a matter of common experience that many of the apparent vagaries of both atmospheric and oceanic flows—for example, fronts, squall lines, polar lows, hurricanes and storm surges—are localized in space and time. Hence during the short term development of such phenomena, the spatial region of interest is a limited area of a much larger domain. The prediction of the flow field within the limited area poses a mixed initial-boundary value (IBV) problem. Some subset of the lateral boundaries of the limited area comprise open (purely geometric) fixed surfaces immersed in the fluid. The IBV problem is highlighted in the practical task of constructing regional weather prediction models and storm surge models for specific coastal regions.

Examples of numerical IBV experiments conducted with finite-difference representation of the linear advection equations, the barotropic vorticity equation, the nonlinear shallow water equations and the full primitive equations of meteorology, are given in [1–5]. The results of these particular experiments indicate that incorrect or over-specification of the lateral boundary conditions leads to spurious oscillations that propagate into the interior of the region. The growth of these oscillations may eventually mask the development of the true solution. In these models, the spurious oscillations were either effectively suppressed by the application of filters or severe smoothing near the boundaries, or the region of integration was taken to be so large as to prevent the oscillations reaching the central region of interest within the desired forecast time.

A probable source for these oscillations was pinpointed by Charney *et al.* [2] in their seminal paper on the integration of the barotropic vorticity equation. On the basis of the observation that conservative advective quantities impose constraints on the determination of the flow variables at outflow lateral boundaries, they suggested that only a minimum subset of the dependent variables (only the normal velocity in their case) should be externally specified at outflow points. Charney [6] also applied the argument to the shallow water equations, the so-called barotropic primitive equations, and concluded that the normal velocity should be externally specified everywhere on the boundary and potential vorticity at inflow points.

Furthermore, in [6] a successful integration of the barotropic primitive equations was undertaken with these lateral boundary conditions. Other successful integrations have been obtained with the barotropic vorticity equation [7], the linear shallow water equations [8], and the barotropic primitive equations [9]. Again, in these studies only a minimum specification of variables was made at the lateral boundaries. To avoid external overspecification at outflow boundary points, the values of certain dependent variables must be ascertained from the interior values and the specified boundary variables. At outflow points these authors adopted ad hoc variations of the Lagrangian advection scheme (upstream difference in space and forward difference in time) for the conservative quantity or quantities. A criterion for assessing the usefulness of this scheme was the a posteriori verification of the "smooth" development of the computed flow.

Studies of the linear advection equation [10; 11, p. 137] indicate forcibly that consistency of the finite difference representation with the continuous equations is not the only criterion governing the choice of schemes at outflow points. These analyses spotlight an aspect of the IBV problem associated with the difference equations, viz these equations may require more boundary conditions than are necessary for the corresponding differential equations. The formal development of a theory of difference approximations for the IBV problem is now gathering momentum (see [12] and the references therein).

In this paper we apply the energy method to certain conventional difference approximations of the IBV problem posed by the linear advection equation, the barotropic vorticity equation, and the nonlinear shallow water equations in *open* systems. The technique of using the energy method to establish suitable boundary conditions was advocated by Morton [13]. This suggestion was persued by Campbell [14] in a stability analysis of a difference scheme for the two dimensional Navier–Stokes equations in a closed system, while Elvius and Sundstrom [15] have recently examined the IBV problem posed by the linearised shallow water equations in an *open* system.

The present work bears the same spirit and tenor as that of Elvius and Sundstrom. In Section II, we outline the particular character of the mathematical problem under consideration. The discussion is developed from a study of the energy method applied to the differential equations and then extended in the succeeding section to the difference equations. We derive conditions governing the boundedness of the solutions of the difference equations and hence suggest suitable formulations for the lateral boundary conditions. Results of numerical experiments conducted with the derived, and other, boundary conditions, are reported in Section IV.

II. Outline of the Mathematical Problem

We study three mixed initial-boundary value problems.

A. *The Linear Advection Equation*

$$(\partial u/\partial t) + c\,\partial u/\partial x = 0, \tag{1a}$$

on the line segment $0 \leqslant x \leqslant 1$ with $0 \leqslant t \leqslant \tau$

Initial conditions are,

$$u(x, 0) = f(x), \qquad 0 \leqslant x \leqslant 1$$

Boundary conditions are

$$u(0, t) = g(t), \qquad 0 \leqslant t \leqslant \tau$$

and c is a positive constant.

B. *The Barotropic Vorticity Equation*

$$(\partial/\partial t)(\nabla^2\psi) + J(\psi, \nabla^2\psi + f) = 0 \tag{1b}$$

where $\psi = \psi(x, y, t)$ is the stream function, $f = f_0 + \beta y$ is the Coriolis parameter, and ∇^2 and J denote the two-dimensional Laplacian and Jacobian, respectively.

Initial conditions are

$$\psi(x, y, 0) = \psi^0(x, y)$$

in the area S given by $(0 \leqslant x \leqslant L, 0 \leqslant y \leqslant W)$.

Boundary conditions are:

normal velocity $\mathbf{v} \cdot \mathbf{n}$ $(=\partial\psi/\partial S)$ be specified everywhere on curve C enclosing S;

vorticity $\nabla^2\psi$ be specified at inflow points $(\mathbf{v} \cdot \mathbf{n} < 0)$

C. *The Shallow Water Equations*

$$(\partial/\partial t)\,\mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} + f(\mathbf{k} \wedge \mathbf{v}) = -g\nabla h$$
$$(\partial/\partial t)\,h + \nabla \cdot (h\mathbf{v}) = 0$$

where $(\mathbf{v}, h)$ denote the horizontal velocity vector and the height of the free surface of the flow of an inviscid, incompressible, and homogeneous fluid. Here g is the gravitational force (assumed constant) and ∇ a horizontal vector operator.

Initial conditions are

$$\mathbf{v}(x, y, 0) = \mathbf{v}^0(x, y)$$

and

$$h(x, y, 0) = h^0(x, y)$$

for the areas given by $(0 \leqslant x \leqslant L, 0 \leqslant y \leqslant W)$.

Boundary conditions are the following.

We consider two different sets of conditions for this case.

Set A: Normal velocity $(\mathbf{v} \cdot \mathbf{n})$ specified everywhere on curve C enclosing S, with the tangential velocity and height field specified at inflow points.

Set B: A variable $\{\mathbf{v} \cdot \mathbf{n} - 2(gh)^{1/2}\}$ specified everywhere on C and the tangential velocity specified at inflow points.

We comment on the two sets of conditions later but first we apply the energy integral method to Eqs. (1) to obtain

$$\frac{\partial}{\partial t}\int_0^1 \tfrac{1}{2}u^2\,dx = -\tfrac{1}{2}c[u^2(1, t) - u^2(0, t)] \tag{2a}$$

$$\frac{\partial}{\partial t}\iint_S \tfrac{1}{2}(\nabla^2\psi)^2\,dS = -\oint_C [(\mathbf{v} \cdot \mathbf{n})\,\tfrac{1}{2}(\nabla^2\psi)^2]\,dC \tag{2b}$$

$$\frac{\partial}{\partial t}\iint_S \tfrac{1}{2}\{h(u^2 + v^2) + gh^2\}\,dS = -\oint_C [(\mathbf{v} \cdot \mathbf{n})\{\tfrac{1}{2}h(u^2 + v^2) + gh^2\}]\,dC \tag{2c}$$

Equations (2) indicate that for each of these problems there exists a quadratic quantity that is increased (decreased) only by inflow (outflow) across the boundary of the region.

To establish the uniqueness of solutions of these systems we can consider two solutions of the basic system and apply the energy integral method to the equation(s) determining the time development of the perturbation, or difference, between the two solutions. This method was applied by Sundstrom to the barotropic vorticity equation [16] and the linearized shallow water equations [17], while the author [18] applied the technique to the nonlinear system C and the baroclinic primitive equations.

In the latter study the author suggested that the "set *A*" boundary conditions constitute a suitable set of lateral boundary conditions for system C. These boundary conditions yield a sufficient condition for uniqueness of a solution provided the velocity field possesses continuous derivatives. It is both pertinent and salutary to note that the proof of uniqueness is based on the assumption of the existence of the solution, and that the height field of the solution need not be continuous.

In contradistinction it is shown in the appendix that the uniqueness of solutions of system *C* can also be established for the "set *B*" boundary conditions provided *both* the velocity and height fields possess continuous derivatives. (This result is a generalization of that obtained in [17] for the linear shallow water equations.) Clearly the set B boundary conditions are preferable on both mathematical and physical grounds. The conditions avoid a pseudo-over specification at inflow for almost all geophysical flow situations, and they also improve the treatment of outgoing gravity waves since they permit characteristic-type equations to be applied at both inflow and outflow boundary points. A comparison of the numerical experiments conducted with these two sets is made in the last section.

The development of a satisfactory 'energy method' theory for difference approximations to our three systems would entail deriving analog difference relationships to the differential relationships (2). Analogs of Eqs. (2) would express the boundedness of a suitable norm of the difference solution, while analogs of the perturbation energy integrals (e.g., Eq. (A3) in the Appendix) would be required to prove the computational stability of the difference schemes. Convergence of the schemes would be yet a further problem.

We confine our attention primarily to the study of the boundedness of suitable norms of our difference systems. The main burden of the analysis is the design of consistent difference formulations at outflow boundary points that ensure that in the difference analogs to Eqs. (2) there is no spurious increase in the norm. We note that it is only the linear system A that boundedness automatically implies stability and convergence. However for low Mach number, closed IBV, flow simulations there is strong evidence [19, 20] that a difference scheme satisfying a boundedness condition inhibits the particularly virulent form of nonlinear computational instability first isolated by Phillips [21].

Thus it is natural for us to choose schemes of this genre in our study. These schemes were developed by Arakawa [22] and Lilly [23], and their use has been principally confined to long term integrations of global atmospheric and oceanic circulations and not to limited area models. However, in the field of regional weather prediction it is envisaged that global, coarse mesh models will provide the boundary data for regional, fine mesh models. Thus limited area models may also be of this type in the future since a high degree of model–model compatibility may prove to be desirable.

To facilitate the description of the difference equations used in this study we adopt the following notational scheme for sum and difference operators,

$$\bar{\varphi}^{x} = \tfrac{1}{2}\{\varphi(x + \tfrac{1}{2}\Delta l) + \varphi(x - \tfrac{1}{2}\Delta l)\} \tag{3}$$

$$\delta_x \varphi = \varphi(x + \tfrac{1}{2}\Delta l) - \varphi(x - \tfrac{1}{2}\Delta l) \tag{4}$$

With (Δl) and (Δt) denoting the discrete spatial and temporal difference increments, our difference approximations to the flux forms of Eqs. (1) take the following form

System A.

$$\delta_t \bar{u}^t = -(c\,\Delta t/\Delta l)\,\delta_x \bar{u}^x. \tag{5}$$

The difference scheme is applied at every interior grid point of a mesh of uniform spacing that spans the range of x, such that

$$x = i(\Delta l), \qquad i = 0, I \qquad \text{and} \quad I(\Delta l) = 1$$

System B.

$$\delta_t \bar{\eta}^t = -(\Delta t/\Delta l)\{\delta_x(\bar{u}^x \bar{\eta}^x) + \delta_y(\bar{v}^y \bar{\eta}^y)\} \tag{6}$$

where

$$u = -\delta_y \bar{\psi}^y, \qquad v = \delta_x \bar{\psi}^x$$

and $\eta = (\delta_{xx}\psi + \delta_{yy}\psi) + f$, is the absolute vorticity. This equation is applied at every interior grid point of an uniform rectangular mesh spanning the flow region with

$$x = (i - 1/2)\,\Delta l, \qquad i = 1, I$$
$$y = (j - 1/2)\,\Delta l, \qquad j = 1, J \text{ and } (I - 1)\,\Delta l = L, (J - 1)\,\Delta l = W.$$

System C.

$$\delta_t \overline{hu}^t = -(\Delta t/\Delta l)\{\delta_x(\overline{hu}^x \bar{u}^x) + \delta_y(\overline{hv}^y \bar{u}^y)\} + 2\Delta t\{fhv - \tfrac{1}{2}(\Delta l)^{-1} gh\, \delta_x \bar{h}^x\}, \tag{7a}$$

$$\delta_t \overline{hv}^t = -(\Delta t/\Delta l)\{\delta_x(\overline{hu}^x \bar{v}^x) + \delta_y(\overline{hv}^y \bar{v}^y)\} - 2\Delta t\{fhu + \tfrac{1}{2}(\Delta l)^{-1} gh\, \delta_y \bar{h}^y\}, \tag{7b}$$

$$\delta_t \bar{h}^t = -(\Delta t/\Delta l)\{\delta_x \overline{hu}^x + \delta_y \overline{hv}^y\}. \tag{8}$$

Again these difference equations are applied at every interior grid point of a uniform rectangular mesh with

$$x = i(\Delta l), \qquad i = 0, I$$
$$y = j(\Delta l), \qquad j = 0, J \qquad \text{and} \quad (I - 1)\,\Delta l = L, (J - 1)\,\Delta l = W.$$

Equation (5) is the usual centred time and space difference approximation to the linear advection equation. Arakawa [20, 22] and Lilly [23] have conducted theoretical and numerical studies of the difference schemes of systems B and C. System C is the barotropic primitive equation analog of the scheme used by Smagorinsky *et al.* [17] and others in studying global scale baroclinic systems.

III. Derivation of Boundedness Conditions

The basic ideas and philosophy of the so-called energy method of analyzing the properties of difference schemes has been expounded elsewhere [11], and will not be presented here. We simply note that hereafter the term "energy" will refer to some specified norm of the solution vector of the particular difference scheme under consideration. The three systems will be treated separately.

System A.

An examination of this system with the energy method is a useful prelude to the study of the other two systems, and indeed averts the proleptic adoption of a boundary scheme proposed in [11, p. 140] in a similar treatment of the same problem.

We write Eq. (5) in the form,

$$u_i^{n+1} = u_i^{n-1} - \alpha\, \Delta_0 u^n, \tag{9}$$

where

$$\alpha = (c\,\Delta t/\Delta l), \qquad \text{and} \qquad \Delta_0 u^n = u_{i+1}^n - u_{i-1}^n,$$

and define an energy S^n by,

$$S^n = \| u^n \|^2 + \| u^{n+1} \|^2 + \alpha\langle u^{n+1}, \Delta_0 u^n\rangle - \alpha(u_I{}^n u_{I-1}^{n+1} - u_0{}^n u_1^{n+1}) - (1/2)\{(u_{I-1}^n)^2 + (u_{I-1}^{n+1})^2\} \tag{10}$$

where

$$\langle \varphi, \psi\rangle = \sum_{i=1,I-1} \varphi_i \psi_i\,,$$

and

$$\| \varphi \|^2 = \langle \varphi, \varphi\rangle.$$

It can then be deduced that

$$\begin{aligned} S^n - S^{n-1} = & -\alpha\langle u^n, \Delta_0 u^{n-1}\rangle - \alpha\langle u^{n-1}, \Delta_0 u^n\rangle \\ & - \alpha(u_I{}^n u_{I-1}^{n+1} - u_I^{n-1} u_{I-1}^n) + \alpha(u_0{}^n u_1^{n+1} - u_0^{n-1} u_1{}^n) \\ & - \tfrac{1}{2}\{(u_{I-1}^{n+1})^2 - (u_{I-1}^{n-1})^2\}, \end{aligned}$$

and using the appropriate summation by parts formula this may be reduced to

$$S^n - S^{n-1} = -\alpha u_I{}^n(u_{I-1}^{n+1} + u_{I-1}^{n-1}) + B^n - \tfrac{1}{2}\{(u_{I-1}^{n+1})^2 - (u_{I-1}^{n-1})^2\}, \qquad (11)$$

where $B^n = \alpha u_0{}^n(u_1^{n+1} + u_1^{n-1})$ denotes the contribution to the change in $S^n - S^{n-1}$ arising from the inflow boundary source.

We now choose to represent the value of $u_i{}^n$ at the boundary point $i = I$ by

$$u_I{}^n = u_{I-1}^{n+1} + u_{I-1}^{n-1} - u_{I-2}^n. \qquad (12)$$

Eliminating u_{I-2}^n from the form of Eq. (9) at point $i = I - 1$ and Eq. (12), yields

$$2\alpha u_I{}^n = -(1 - \alpha)\, u_{I-1}^{n+1} + (1 + \alpha)\, u_{I-1}^{n-1}.$$

Substituting this relationship into Eq. (11) leads to

$$S^n - S^{n-1} = -(1/2)\, \alpha(u_{I-1}^{n+1} + u_{I-1}^{n-1})^2. \qquad (13)$$

We now note the inequality

$$\langle u^{n+1}, \Delta_0 u^n\rangle = \| u^{n+1} \|^2 + \| u^n \|^2 + u_I{}^n u_{I-1}^{n+1} - u_0{}^n u_1^{n+1} - (1/2)\{(u_{I-1}^n)^2 + (u_{I-1}^{n+1})^2\}.$$

Hence it follows from Eq. (10) that

$$(1 - \alpha)(\| u^{n+1} \|^2 + \| u^n \|^2) \leqslant S^n \leqslant (1 + \alpha)(\| u^{n+1} \|^2 + \| u^n \|^2). \qquad (14)$$

Thus from relations (13) and (14) we deduce that

$$\| u^n \|^2 \leqslant (1 - \alpha)^{-1} \Big\{(1 + \alpha)(\| u^1 \|^2 + \| u^0 \|^2) + \sum_{N=1,n} (A^N + B^N)\Big\} \qquad (15)$$

where

$$A^n = -(1/2)\, \alpha(u_{I-1}^{n+1} + u_{I-1}^{n-1})^2.$$

Inequality (15) is the nearest finite difference analog we seek to the corresponding differential relationship, Eq. (2a). For $\alpha < 1$ it shows that the square of the norm, $\| u^n \|^2$, is bounded by its initial values, i.e., $(\| u^1 \|^2 + \| u^0 \|^2)$, and the difference between the boundary source terms, B^N, and the sink terms A^N.

This concludes our proof of boundedness for System A, but the following comments are also pertinent to the discussion of the IBV problem. The difference equation at the boundary mesh point is given by

$$u_{I-1}^{n+1} = u_{I-1}^{n-1} - \alpha(u_{I-1}^{n+1} + u_{I-1}^{n-1} - 2u_{I-2}^{n}). \tag{16}$$

This is a consistent difference approximation to the differential equation at the open boundary. Moreover since the difference Eq. (9) can support solutions on two independent lattices for a purely initial value problem, it is natural in the IBV problem to consider a scheme with the form of Eq. (16) to prevent coupling of these lattices at the outflow boundary.

In the next section we indicate that a failure to satisfy the consistency conditions at the outflow point permits the growth of spurious oscillations at the boundary. We also note that, unlike the second order scheme of Eq. (9), the difference Eq. (16) is correct only to order (Δl) and corresponds to adding an effective viscosity term to the differential equation with the viscosity coefficient taking the value

$$(1/2)\, c(\Delta l)(1 - \alpha^3).$$

Gustafsson [24] has, however, justified the adoption of a difference formulation on the boundary with an accuracy that is one order lower than that for the interior scheme.

System B.

The inner product of Eq. (6) with $(\eta^{n+1} + \eta^{n-1})$ yields

$$\| \eta^{n+1} \|^2 - \| \eta^{n-1} \|^2 = -\mu\langle(\eta^{n+1} + \eta^{n-1}), \gamma^n\rangle$$

where

$$\gamma^n = \{\delta_x(\overline{u^n}^x, \overline{\eta^n}^x) + \delta_y(\overline{v^n}^y, \overline{\eta^n}^y)\},$$
$$\mu = 2\Delta t/\Delta l,$$

and now

$$\langle\varphi, \psi\rangle = \sum_{i=1,I-1} \sum_{j=1,J-1} \varphi_{ij}\psi_{ij}$$

with

$$\| \varphi \|^2 = \langle\varphi, \varphi\rangle.$$

We define an energy by

$$S^n = \| \eta^{n+1} \|^2 + \| \eta^n \|^2 + \mu\langle\eta^{n+1}, \gamma^n\rangle + \mu \sum_{i,j} (A^n + B^n), \tag{17}$$

where $\sum_{i,j} A^n$ and $\sum_{i,j} B^n$ denotes the contribution to S^n from outflow and inflow boundary points, respectively. The component of $\sum_{i,j} A^n$ at a point $(I-1, j)$ has the form

$$A^n\,|_{I-1,j} = -(1/4)\,\eta_I{}^n \eta_{I-1}^{n+1}(u_I{}^n + u_{I-1}^n),$$

with similar contributions from the other outflow boundary points. It follows from (17) and the summation by parts formula that

$$S^n - S^{n-1} = -\mu\langle \eta^{n-1}, \gamma^n\rangle - \mu\langle \eta^n, \gamma^{n-1}\rangle + \mu \sum_{i,j} (A^n - A^{n-1} + B^n - B^{n-1}). \tag{18}$$

To proceed further we require some inequality relations for the inner product term in Eqs. (17) and (18). For clarity we develop below only the x-component of these relations.

$$\begin{aligned}\langle \eta^{n+1}, \gamma^n\rangle_{x\,\mathrm{comp}} &= \langle \eta^{n+1}, \{u^{n\,xx}\,\delta_x\overline{\eta^n}^{x} + \delta_x\overline{u^n}^{x}\,\overline{\eta^n}^{xx}\}\rangle \\ &= \langle \eta^{n+1}, \{(1/2)(\overline{u^n}^{2x} + u)\,\delta_x\overline{\eta^n}^{x} + \delta_x\overline{u^n}^{x}((1/2)\,\overline{\eta^n}^{x} + \eta^n)\}\rangle.\end{aligned} \tag{19}$$

Here we have used successively the following operator identities,

$$\delta_x(\overline{\varphi}^{x}\overline{\psi}^{x}) \equiv \overline{\varphi}^{xx}\,\delta_x\overline{\psi}^{x} + \overline{\psi}^{xx}\,\delta_x\overline{\varphi}^{x},$$

and

$$\overline{\varphi}^{xx} \equiv (1/2)(\overline{\varphi}^{2x} + \varphi).$$

Now since,

$$\delta_x\overline{u^n}^{x} = -\delta_y\overline{v^n}^{y},$$

we omit the last term in Eq. (19) and in the counterpart term in the y component equation. The remaining terms are rearranged in the form,

$$\langle \eta^{n+1}, \gamma^n\rangle_{x\,\mathrm{comp}} = \sum_{i,j} (1/4)\,\eta_{ij}^{n+1}\{u_{i+1,j}^n\eta_{i+1,j}^n + u_{ij}(\eta_{i+1,j}^n - \eta_{i-1,j}^n) - u_{i-1,j}^n\eta_{i-1,j}^n\}.$$

Application of the Cauchy–Schwartz inequality then leads to

$$\langle \eta^{n+1}, \gamma^n\rangle_{x\,\mathrm{comp}} \leqslant (1/2)\,|\,V\,|\,(\|\,\eta^{n+1}\,\|^2 + \|\,\eta^n\,\|^2) + \sum_{i,j} (A^n + B^n), \tag{20}$$

where

$$|\,V\,| = \max_{i,j}(|\,u_{ij}\,|, |\,v_{ij}\,|).$$

Similar tedious algebraic manipulations yield the inequality

$$-\langle \eta^n, \gamma^{n-1} \rangle_{x\,\mathrm{comp}} - \langle \eta^{n-1}, \gamma^n \rangle_{x\,\mathrm{comp}} \leqslant (1/2)\,|\,\Delta^n V\,|\,(\|\,\eta^n\,\|^2 + \|\,\eta^{n-1}\,\|^2 + \sum (A'^n - B'^n), \tag{21}$$

where

$$|\,\Delta^n V\,| = \max_{i,j,N}(|\,u_{ij}^N - u_{ij}^{N-1}\,|, |\,v_{ij}^N - v_{ij}^{N-1}\,|),$$

and $\sum A'^n$ and $\sum B'^n$ retain their usual notational interpretations, with

$$A'^n\,|_{I-1,j} = -(1/4)(u_{I-1}^n \eta_{I-1}^{n-1} \eta_I{}^n + u_{I-1}^{n-1} \eta_{I-1}^n \eta_I^{n-1} + u_I{}^n \eta_I^{n-1} \eta_{I-1}^n + u_I{}^n \eta_I{}^n \eta_{I-1}^{n-1}).$$

Inserting inequality (21) and the y-component counterpart inequality in Eq. (18) leads to

$$S^n - S^{n-1} \leqslant |\,\Delta^n V\,|\,(\|\,\eta^n\,\|^2 + \|\,\eta^{n-1}\,\|^2) + \mu \sum_{i,j} (A^{*n} + B^{*n}), \tag{22}$$

where

$$A^{*n} = A^n - A^{n-1} + A'^n,$$
$$B^{*n} = B^n - B^{n-1} + B'^n,$$

with

$$A^{*n}\,|_{I-1,j} = -(1/4)(u_I{}^n + u_{I-1}^n)\,\eta_I{}^n(\eta_{I-1}^{n+1} + \eta_{I-1}^{n-1}). \tag{23}$$

Moreover using inequality (20) with Eq. (17), we deduce that

$$(1 - \alpha)(\|\,\eta^{n+1}\,\|^2 + \|\,\eta^n\,\|^2) \leqslant S^n \leqslant (1 + \alpha)(\|\,\eta^{n+1}\,\|^2 + \|\,\eta^n\,\|^2), \tag{24}$$

where

$$\alpha = \mu\,|\,V\,|\,.$$

From inequalities (22) and (24) we obtain the boundedness relation

$$\|\,\eta^n\,\|^2 \leqslant (1 - \alpha)^{-1} \Big\{(1 + \alpha)(1 + \epsilon)^n\,(\|\,\eta^1\,\|^2 + \|\,\eta^0\,\|^2) + \sum_N \mu\epsilon^{N-n} \Big(\sum_{i,j} [A^{*n} + B^{*n}]\Big)\Big\} \tag{25}$$

where

$$\epsilon = [1 + (1 - \alpha)^{-1} \max_N(|\,\Delta^n V\,|)].$$

Inequality (25) shows that the difference scheme is suitably bounded provided

$$\alpha < 1 \qquad \text{and} \qquad \left\{\sum_N \mu\epsilon^{N-n}\left(\sum_{i,j} A^{*n}\right)\right\} < 0.$$

The first condition is of purely practical significance, while it follows from Eq. (23) that the second will be satisfied at outflow points if

$$\eta_I{}^n(\eta_{I-1}^{n+1} + \eta_{I-1}^{n-1}) > 0 \tag{26}$$

when

$$(u_I{}^n + u_{I-1}^n) > 0. \tag{27}$$

We again adopt the approximation

$$\eta_I{}^n = \eta_{I-1}^{n+1} + \eta_{I-1}^{n-1} - \eta_{I-2}^n,$$

and we note that it yields a consistent difference approximation to Eq. (16) at the first interior mesh point.

The normal velocity is to be specified a priori, and hence relation (27) is satisfied when the boundaries are placed midway between the mesh points. This particular location of the boundaries implies the Poisson equation for the stream function must be solved subject to mixed Dirichlet–Neumann boundary conditions.

We do not attempt rigorously to establish the validity of (26). However, we note that its validity is readily monitored in a computation. It is also helpful to examine the nature of the physical problem under consideration. The barotropic vorticity equation, Eq. (16), is a useful means of representing large scale transient features of the atmospheric circulation, and on this spatial scale the absolute vorticity, η, is generally positive. Thus we anticipate that relation (26) will usually be satisfied if the scale of the vorticity pattern is much larger than the mesh length throughout the integration period.

System C

Suitable inner product operations on Eqs. (7) and (8) yield

$$\| U^{n+1} \|^2 - \| U^{n-1} \|^2$$
$$= -\mu\langle(U^{n+1} + U^{n-1}), \gamma^n\rangle + \langle(U^{n+1} + U^{n-1}), \{\omega V^n - \mu\varphi^n\, \delta_x\overline{\varphi^n}^x\}\rangle$$

$$\| V^{n+1} \|^2 - \| V^{n-1} \|^2$$
$$= -\mu\langle(V^{n+1} + V^{n-1}), \theta^n\rangle - \langle(V^{n+1} + V^{n-1}), \{\omega U^n + \mu\varphi^n\, \delta_y\overline{\varphi^n}^y\}\rangle$$

$$\| \varphi^{n+1} \|^2 - \| \varphi^{n-1} \|^2$$
$$= -\mu(\langle\varphi^{n+1} + \varphi^{n-1}), \sigma^n\rangle$$

where

$$\varphi = gh, \qquad U = \varphi u, \qquad V = \varphi v,$$
$$\mu = 2\Delta t/\Delta l, \qquad \omega = 2(\Delta t) f$$

and

$$\gamma^n = \delta_x(\overline{U}^{x}\overline{u}^{x}) + \delta_y(\overline{V}^{y}\overline{u}^{y}),$$
$$\theta^n = \delta_x(\overline{U}^{x}\overline{v}^{x}) + \delta_y(\overline{V}^{y}\overline{v}^{y}),$$
$$\sigma^n = \delta_x(\overline{U}^{x}) + \delta_y(\overline{V}^{y}).$$

We define the energy S^n,

$$\begin{aligned} S^n = {} & \| U^{n+1} \|^2 + \| U^n \|^2 + \| V^{n+1} \|^2 + \| V^n \|^2 + c^2\{\| \varphi^{n+1} \|^2 + \| \varphi^n \|^2\} \\ & + \mu\langle U^{n+1}, \gamma^n\rangle + \mu\langle U^{n+1}, \varphi^n\, \delta_x\overline{\varphi^n}^{x}\rangle - \omega\langle U^{n+1}, V^n\rangle \\ & + \mu\langle V^{n+1}, \theta^n\rangle + \mu\langle V^{n+1}, \varphi^n\, \delta_y\overline{\varphi^n}^{y}\rangle + \omega\langle V^{n+1}, U^n\rangle \\ & + \mu c^2\langle \varphi^{n+1}, \sigma^n\rangle + \mu \sum_{i,j} (A^n + B^n). \end{aligned} \tag{28}$$

Again, $\sum A^n$ and $\sum B^n$ retain their usual notational significance with

$$\begin{aligned} A^n |_{I-1,j} = {} & -(1/4)\{U_{I-1}^{n+1} u_I{}^n (U_I{}^n + U_{I-1}^n) + V_{I-1}^{n+1} v_I{}^n (U_I{}^n + U_{I-1}^n) \\ & + U_{I-1}^{n+1} u_{I-1}^n U_I{}^n + V_{I-1}^{n+1} v_{I-1}^n V_I{}^n + 2U_{I-1}^{n+1} \varphi_{I-1}^n U_I{}^n + 4c^2 \varphi_{I-1}^{n+1} U_I{}^n\}, \end{aligned}$$

and $c^2 = gH$, the value of the height H being left unspecified for the moment. From Eq. (23) it can be deduced that

$$\begin{aligned} & (1 - \alpha)(\| U^{n+1} \|^2 + \| U^n \|^2 + \| V^{n+1} \|^2 + \| V^n \|^2 + c^2\{\| \varphi^{n+1} \|^2 + \| \varphi^n \|^2\}) \\ & \quad \leqslant S^n \\ & \quad \leqslant (1 + \alpha)(\| U^{n+1} \|^2 + \| U^n \|^2 + \| V^{n+1} \|^2 + \| V^n \|^2 + c^2\{\| \varphi^{n+1} \|^2 + \| \varphi^n \|^2\}) \end{aligned} \tag{29}$$

where

$$\alpha = \{\mu \max_{i,j,N}(| u_{ij}^N |, | v_{ij}^N |) + \mu \max_{i,j,N}(c^2, | \varphi_{ij}^N |) + (1/2)\,\omega\}$$

And, furthermore,

$$\begin{aligned} S^n - S^{n-1} \leqslant {} & \Delta^*(\| U^{n+1} \|^2 + \| U^n \|^2 + \| V^{n+1} \|^2 + \| V^n \|^2 + c^2\{\| \varphi^{n+1} \|^2 + \| \varphi^n \|^2\}) \\ & + \mu \sum_{i,j} (A^{*n} + B^{*n}) \end{aligned} \tag{30}$$

with

$$A^{*n}|_{I-1,j} = -(1/4)\{2\varphi_I{}^n\varphi_{I-1}^n(U_{I-1}^{n+1} + U_{I-1}^{n-1}) + 2c^2U_I{}^n(\varphi_{I-1}^{n+1} + \varphi_{I-1}^{n-1})$$
$$+ [U_I{}^n(u_I{}^n + u_{I-1}^n) + u_I{}^nU_{I-1}^n](U_{I-1}^{n+1} + U_{I-1}^{n-1})$$
$$+ [U_I{}^n(v_I{}^n + v_{I-1}^n) + v_I{}^nU_{I-1}^n](V_{I-1}^{n+1} + V_{I-1}^{n-1})\}, \quad (31)$$

and Δ^* is the now customary positive definate maximum of some subset of the difference variables.

The abysmal algebra required to establish inequalities (29) and (30) is not reproduced here. As for system B, these inequalities suffice to show that our difference scheme is suitably bounded provided $\alpha < 1$ and the outflow term, $A^{*n}|_{I-1,j}$ is negative definite. The first condition effectively constraints the unpsecified height H to a range such that $0 < gH \leqslant \max_{i,j,N}(\varphi_{ij})$. To examine $A^{*n}|_{I-1,j}$ we must introduce our computational outflow conditions.

It was shown in the appendix that the set B lateral boundary conditions are sufficient conditions for the uniqueness of solutions of the continuous equations and are also inter alia the lateral boundary conditions we must endeavor to satisfy with the difference equations. In practice there may be operational, numerical or computational constraints that militate against the strict adherence to the minimum subset specification but any overspecification must be examined with caution. For the problem at hand, the results for system A can be readily extended to cover the linear equations of system C written in characteristic form. However, the set B boundary conditions are not amenable to theoretical examination with our choice of difference representation for the nonlinear equations. We are thus forced to resort to an investigation of the outflow boundary points with the set A conditions, and concomitantly acknowledge that spurious effects might be generated at inflow for a nontrivial overspecification of the dependent variables.

We adopt the following approximations at outflow,

$$U_I{}^n = U_{I-1}^{n+1} + U_{I-1}^{n-1} - U_{I-2}^n,$$
$$v_I{}^n = v_{I-1}^{n+1} + v_{I-1}^{n-1} - v_{I-2}^n;$$

and then regroup the terms in (31):

$$A^{*n}|_{I-1,j} = -(1/4)(U_{I-1,j}^{n+1} + U_{I-1,j}^{n-1})\{2\varphi_{I,j}^n\varphi_{I-1,j}^n + U_{I,j}^n u_{I,j}^n\}$$
$$- (1/4)\, U_{I,j}^n\{2c^2(\varphi_{I-1,j}^{n+1} + \varphi_{I-1,j}^{n-1}) + u_{I-1,j}^n(U_{I-1,j}^{n+1} + U_{I-1,j}^{n-1})$$
$$+ (v_{I,j}^n + v_{I-1,j}^n)(V_{I-1,j}^{n+1} + V_{I-1,j}^{n-1})\}$$
$$- (1/4)\, U_{I-1,j}^n\{u_{I,j}^n(U_{I-1,j}^{n+1} + U_{I-1,j}^{n-1}) + v_{I,j}^n(V_{I-1,j}^{n+1} + V_{I-1,j}^{n-1})\}$$

If the mesh boundary is placed at $i = (I - 1)$, then $U^n_{I-1,j}$ is specified for all n. When there is outflow, at $(I - 1)$ for three successive time steps, $(n - 1, n, n + 1)$ then, noting that $\phi^n_{i,j}$ is positive, we infer that $A^{*n}|_{I-1,j}$ is certainly negative definite if

$$U^n_{I,j} > 0,$$

and

$$2c^2(\varphi^{n+1}_{I-1,j} + \varphi^{n-1}_{I-1,j}) + u^n_{I-1}(U^{n+1}_{I-1,j} + U^{n-1}_{I-1,j}) + (v^{n+1}_{I-1,j} + v^{n-1}_{I-1,j})(V^{n+1}_{I-1,j} + V^{n-1}_{I-1,j})$$
$$> (V^n_{I-1,j} - V^n_{I-2,j})(V^{n+1}_{I-1,j} + V^{n-1}_{I-1,j}).$$

Order of magnitude considerations indicate that these conditions are usually satisfied in geophysical flow applications if the spatial scale of the motion is much larger than the computational mesh length.

IV. Numerical Experiments

To test the outflow differencing schemes proposed in the previous section three sets of experiments were carried out with the linear advection equation, and the one- and two-dimensional shallow water equations. For comparison, various other outflow schemes were also tested in these sets of experiments.

(a) *Linear Advection Equation*

With the difference scheme of Eq. (9) applied at interior points experiments were undertaken with following outflow schemes:

(1) $u_I{}^n = u^{n+1}_{I-1} + u^{n-1}_{I-1} - u^n_{I-2}$,

(2) $u_I{}^n = (1 - \alpha)\, u_I^{n-1} + \alpha u^{n-1}_{I-1}$,

(3) $u_I{}^n = (1/2)(u^{n+1}_{I-1} + u^{n-1}_{I-1})$,

(4) $u_I{}^n = 2u^n_{I-1} - u^n_{I-2}$.

Gustafsson *et al.* [25] successfully constructed a stability theory for the Lagrangian advection scheme 2 based upon an extension of the Ryabenkii–Godounow condition to IBV problems. Scheme 3 was proposed in [11], but it is salutary to note that it does not satisfy the consistency condition. The linear spatial extrapolation scheme 4 is consistent, but its instability has been demonstrated in [10, 11].

Experiments were undertaken for $\alpha = (0.75, 0.5)$ and $I = 31$ with two initial states comprising a sinusoidal wave of wavenumber 2 and a wedge shaped configuration. Schemes 3 and 4 were unsatisfactory. Both schemes produced large amplitude error fiedls $u'(= u_{\text{correct}} - u_{\text{calculated}})$. Some of the results obtained with schemes 1 and 2 with the sinusoidal wave and with $\alpha = .75$ are depicted in Fig. 1. The initial field, u, and the error field, u', of the computed solution after a

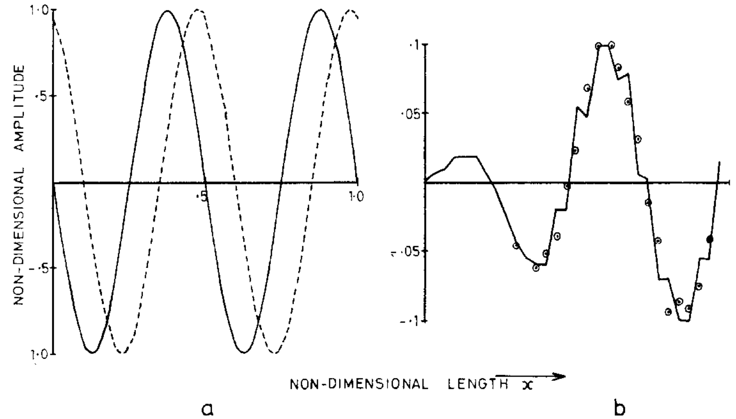

FIG. 1. (a) u field at $t = 0$ (solid line) and $t = 24\ (\Delta t)$ (dashed line); (b) u' field for scheme 1 (solid line) and scheme 2 (○) at $t = 24\ (\Delta t)$.

time $24(\Delta t)$ are shown, together with the corresponding correct u field. A measure of the effect of the boundary formulation as opposed to the errors due solely to the interior differencing scheme is obtained by comparing the u' field in the neighborhood of $x = 0$ and $x = 1$. There is a marked two-grid increment wave in the error field of scheme 1, but there is no other significant difference between the two schemes.

(b) *One dimensional shallow water equations*

The one dimensional form of Eqs. (7) and (8) were integrated for initial quiescent conditions with a hump of water in a limited portion of the flow region and an uniform height field elsewhere. Wall boundary conditions were placed at $i = 0$ and below we present a catalog of the schemes tested at the $i = I - 1$ boundary for time level n.

Boundary Scheme	Difference Formulation
(1a) Set A. Specify:	$u_{I-1}^{n+1}, \phi_{I-1}^{n+1}$ (at inflow),
	$u_{I-1}^{n+1},\ U_I^n = U_{I-1}^{n+1} + U_{I-2}^{n-1} - U_{I-2}^n$ (at outflow),
(1b) Set A. Specify:	$u_{I-1}^{n+1}, \varphi_{I-1}^{n+1}$ (at inflow),
	$u_{I-1}^{n+1},\ \varphi_{I-1}^{n+1} = \varphi_{I-1}^n - \frac{1}{2}u(U_{I-1}^n - U_{I-2}^n)$ (at outflow).

(2) Set B. Specify $(u - 2c)_{I-1}^{n+1}$ and calculate $(u + 2c)_{I-1}^{n+1}$ with a centered time and space approximation of

$$(\partial/\partial t)(u + 2c) = -(u + c)(\partial/\partial x)(u + 2c),$$

and assume

$$(u + 2c)_I^n = (u + 2c)_{I-1}^{n+1} + (u + 2c)_{I-1}^{n-1} - (u + 2c)_{I-2}^n .$$

(3) Buffer Zone: Diffusive terms with the form

$$[\Delta t/(\Delta l)^2]\{2\nu\, \delta_{xx} U^{n-1} + \tfrac{1}{2}\delta_x \overline{\nu}^x\, \delta_x \overline{U^{n-1}}^x\},$$

and

$$[\Delta t/(\Delta l)^2]\{2\nu\, \delta_{xx} \phi^{n-1} + \tfrac{1}{2}\delta_x \overline{\nu}^x\, \delta_x \overline{\phi^{n-1}}^x\},$$

added to Eqs. (7a) and (8), respectively, in the region I $\leqslant i \leqslant I + 8$

Scheme 1a was suggested in the previous section. The semi implicit formulation enables an algorithm based upon Eq. (8) to be developed to compute ϕ_{I-1}^{n+1}. Scheme 1b corresponds to the usual up-stream differencing technique. The third scheme represents a highly viscous buffer zone designed to, at least partially, absorb the energy of incoming gravity waves.

The lateral boundary velocity and height fields required for schemes 1 and 2 were obtained by performing a separate integration over a considerably extended domain in the positive x direction. Moreover the data set from this integration was also adopted as the desired solution in the limited domain and thus formed a basis for the comparison of the various schemes.

Dimensional equations were used in the experiments with $\Delta t = 360$ sec $\Delta l = 10^5\, m$, $I = 38$, and the initial fields given by:

$$u_i^0 = 0 \qquad (i = 0, I - 1),$$

$$\varphi_i^0 = 5.1 + 10^4\,\mathrm{m}^2\, s^{-2}\, (0 \leqslant i \leqslant 10,\ 30 \leqslant i \leqslant 37),$$

$$= 5.1\{1 + 0.2 \sin[(i - 10)\,\pi/20]\} \times 10^4\,\mathrm{m}^2\,\mathrm{sec}^{-2}\ [\text{elsewhere}).$$

Fig. 2 traces the time development of the geopotential height (ϕ) and the error geopotential field ($\phi' = \varphi_{\mathrm{corr.}} - \varphi_{\mathrm{calc.}}$) generated by schemes 1 and 2. The schemes behave satisfactorily with comparatively weak φ' fields. An encouragingly good result was obtained with scheme 2 when the value of $(u - 2c)_{I-1}^n$ was maintained at its initial value corresponding to the quiescent conditions in the far field.

For these particular initial conditions, and with $\nu = 10^6 m^2\,\mathrm{sec}^{-1}$, scheme 3 does

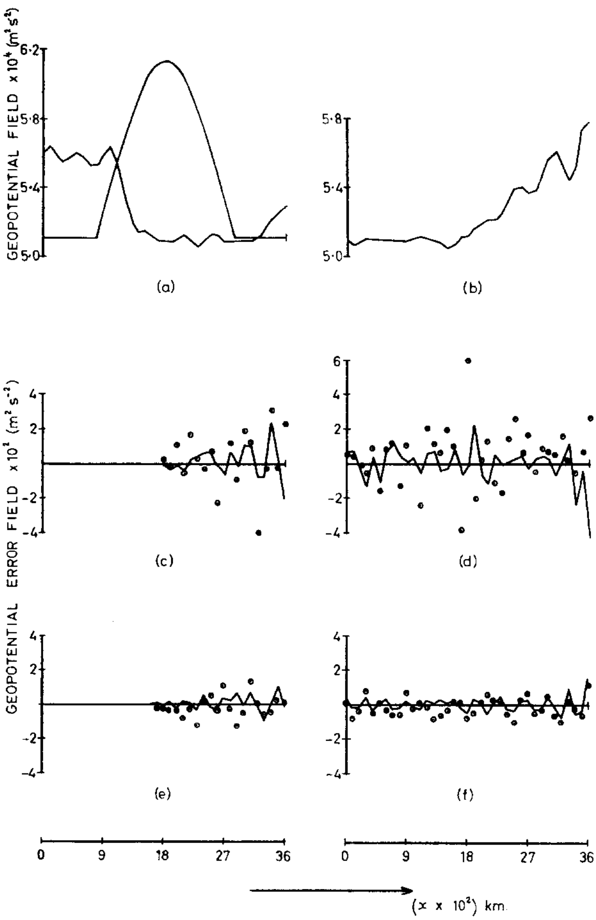

FIG. 2. Development in time of the height field in the one-dimensional shallow water equation. Geopotential field at (a) $t = 0, 30\Delta t$ and at (b) $t = 60\Delta t$. Geopotential error field at (c) $t = 30\Delta t$ and at (d) $t = 60\Delta t$, with scheme 1a (solid line) and scheme 1b (○). Similarly, (e) and (f) depict geopotential error field at the same times with scheme 2 (solid line). The ○ points denote the error field obtained with scheme 2 and the a priori specification of a constant boundary value for $(u - 2c)$.

not succeed in damping the amplitude of the incident gravity waves sufficiently and spurious effects propogate into the region.

(c) *Two-Dimensional Shallow Water Equations*

Experiments were conducted with the full difference Eqs. (7) and (8). The fluid was assumed to be on a β-plane and two radically different types of initialconditions

—a quasinondivergent flow and an almost irrotational flow—were used to test the lateral boundary schemes. The first type corresponded to the β-plane equivalent of the usual Rossby–Haurwitz wave [26] with a uniform zonal flow and a superimposed, large amplitude Rossby wave that drifted steadily eastward remaining virtually unchanged in structure over the integration period. In constrast, the initial state for the second type of flow was a centrally humped free surface in a otherwise quiescent fluid

Lateral boundary conditions at $(I-1, j)$ for time level n were posed as follows:

Boundary Scheme	Difference Specification
1. Set A. Specify	$u_{I-1,j}^{n+1}, \phi_{I-1,j}^{n+1}, v_{I-1,j}^{n+1}$ (at inflow)

Specify $u_{I-1,j}^{n+1}$ and assume

$$U_{I,j}^{n} = U_{I-1,j}^{n+1} + U_{I-1,j}^{n-1} - U_{I-2,j}^{n}$$
$$v_{I,j}^{n} = v_{I-1,j}^{n+1} + v_{I-1,j}^{n-1} - v_{I-2,j}^{n} \qquad \text{(at outflow)}$$

2. Set B. Specify $(u-2c)_{I-1,j}^{n+1}$ calculate $(u+2c)_{I-1,j}^{n+1}$ (at outflow and inflow) with a centred time and space approximation of

$$\frac{\partial}{\partial t}(u+2c) = -(u+c)\frac{\partial}{\partial x}(u+2c) - v\frac{\partial}{\partial y}(u+2c) - c\frac{\partial v}{\partial y} + fv$$

Specify $v_{I-1,j}^{n+1}$ at inflow and calculate $v_{I-1,j}^{n+1}$ at outflow as in scheme A.

Similar treatment enables all other mesh boundary values to be calculated except for the four corner points. If the vector velocity is directed into the flow region at a corner point we specify $\mathbf{v}$ and h, otherwise we specify $\mathbf{v}$ and then to determine $\varphi_{I-1,J-1}^{n+1}$ we assume

$$U_{I,J-1}^{n} = U_{I-1,J-1}^{n+1} + U_{I-1,J-1}^{n-1} - U_{I-2,J-1}^{n},$$

and

$$V_{I-1,J}^{n} = V_{I-1,J-1}^{n+1} + V_{I-1,J-1}^{n-1} - V_{I-1,J-2}^{n}.$$

Once again the required lateral boundary conditions and the desired comparison solution were obtained by performing a separate integration over a larger domain. To set the problem in a meteorological context the north–south and east–west dimensions of the larger domain were chosen to coincide respectively with the pole-equator distance and half the circumference of the earth at midlatitudes. The dimensions of our inner domain and its location relative to the larger domain is indicated schematically in Fig. 3.

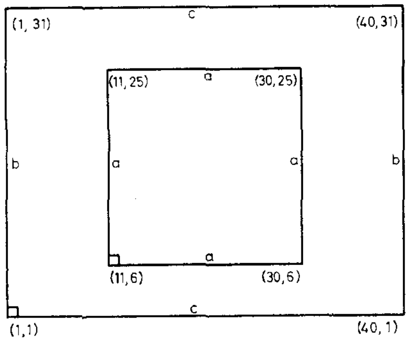

FIG. 3. Schematic diagram of inner and outer integration domains. Grid size indicated at bottom left of each domain. Periodic boundary conditions applied at boundaries 'b' and wall boundaries at '*c*'.

The initial conditions for these experiments when referred to the larger domain are given by,

Type 1.

$$u = U_0 - mA \sin lx \cos my$$

$$v = lA \cos lx \sin my$$

$$\phi = A\{f + m\beta(l^2 + m^2)^{-1}\} \sin lx \cos my + (1/4) A^2(m^2 \cos 2lx + l^2 \cos 2my) - \tfrac{1}{2}\beta U_0 y^2.$$

The Coriolis parameter is given by

$$f = f_0 + \beta y, \text{ with } f_0 = 1 \cdot 3 \times 10^{-4} \text{ sec}^{-1},$$

and U_0 and A represent respectively the uniform velocity of the zonal flow and the amplitude of the Rossby wave. Choosing

$$\Delta l = 3 \cdot 10^5 \text{ m}, \qquad \Delta t = 600 \text{ sec}$$

then

$$l = 2\pi p/38(\Delta l)\, m^{-1}$$

$$m = \pi q/30(\Delta l)$$

with p and q assuming integer values. The pattern corresponds to a Rossby wave with a planetary wavenumber $2p$

Type 2.

$$u \equiv v \equiv 0 \text{ for all } i, j$$

$$\begin{aligned} \varphi/g &= 5200\{1 + (1/5)\sin[\pi(i-15)\Delta l/11]\sin[\pi(j-10)\Delta l/11] \\ &\qquad\qquad\qquad\qquad \text{for } 16 \leqslant i \leqslant 25,\ 11 \leqslant j \leqslant 20, \\ &= 5200 \qquad\qquad\qquad \text{elsewhere.} \end{aligned}$$

Details of the experiments carried out with these two types of initial conditions are presented in Table 1. The time development of RMS error height for the

TABLE 1

Experiments Undertaken with the Shallow Water Equations

		Initial conditions				Boundary	Integration
			$A \times 10^7$	$\beta \times 10^{-11}$		conditions	time
Exp. no.	Type	(m sec^{-1})	(m^2 sec^{-1})	(m^{-1} sec^{-1})	(p, q)	(Set)	(hr)
1a, 1b	1	20	4	1.619	2, 1	A, B	48
2a, 2b	1	20	2	1.619	4, 1	A, B	48
3	1	15	2	0	2, 4	B	48
4a, 4b	2	—	—	—	— —	A, B	8

various experiments are portrayed in Fig. 4. For the experiments conducted with the Set A conditions two curves (t and nt) are drawn corresponding to trivial and nontrivial overspecification of the height field at inflow boundary points. The height field for the (nt) cases was determined by performing a spatial average of the correct (t) values with the formula $\varphi_{nt} = \varphi_t + \mu\nabla^2\varphi_t$ with $\mu = 0.15$,

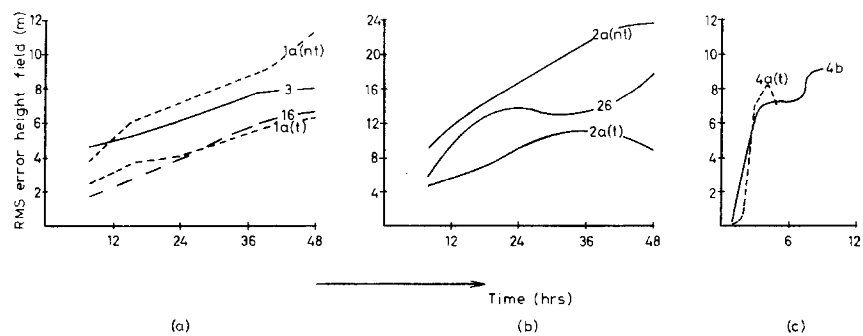

FIG. 4. Development in time of the RMS error height field in the limited domain integrations for the experiments conducted with the two dimensional shallow water equations. These experiments are listed in Table 1.

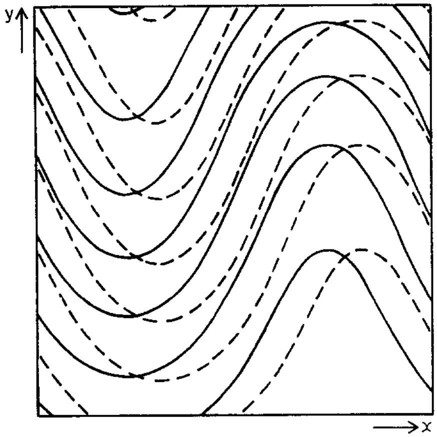

FIG. 5. Height field at $t = 0$ (solid line) and $t = 24$ hr (dashed line) for experiment 1 with the two-dimensional shallow water equations. Contours at intervals of 300 m.

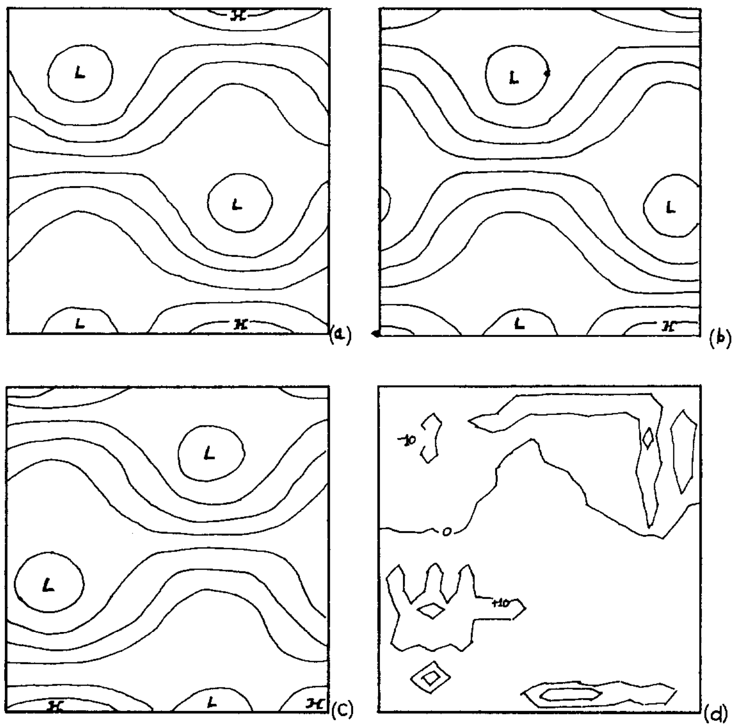

FIG. 6. (a), (b), (c): Height field in the limited domain at $t = 0$, 24, 48 hr, respectively, for experiment 3. Contours at intervals of 150 m. (d) Error height field in the limited domain after 48 hr. Contours at intervals of 10 m.

The controlled behavior of the (*nt*) experimental runs is a hopeful indication that the generalization of the Set A conditions to the baroclinic primitive equations suggested in [18] may be operationally useful. The RMS error velocity fields were of the order of, or less than, 1.0 m sec^{-1} at the termination of all the integrations.

Figures 5 and 6 depict the developemnt of the height field of the fluid in the inner domain during the execution of experiments 1b and 3, together with the final spatial distribution of the error height field. The last figure indicates that comparatively, small scale, closed lows pass successfully into and out from the inner domain during the integration.

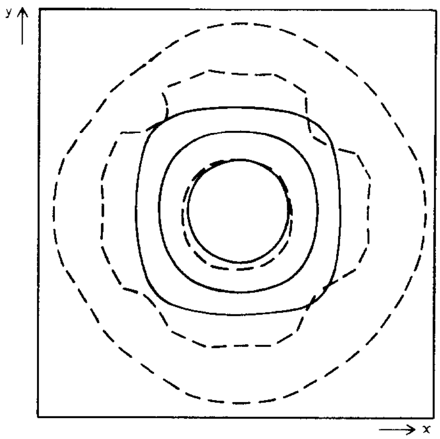

FIG. 7. Height field at $t = 0$ (solid line) and $t = 8$ hr (dashed line) for Type 2 initial conditions. Contours at intervals of 300 m.

V. Conclusions and Further Remarks

The results of the three sets of experiments undertaken are encouraging. They indicate that the semiimplicit boundary differencing technique is an acceptable formulation for the particular interior differencing schemes considered herein. The technique does not appear to engender boundary instabilities. We also conclude that the energy method of analyzing the properties of difference schemes constitutes an useful approach to formulating the boundary difference schemes for the "open" IBV problem.

We have set our treatment of the IBV problem for open systems within a formal mathematical framework. However, it is important to note the practical restrictions that arise from a consideration of the nature of the morphism formed by the real flow and model equations.

We have already indicated that for typical geophysical flow applications the boundary data for the limited area models will be acquired from larger models with coarser mesh lengths. Thus there will be errors in the specified boundary data arising from the differing numerical properties of the two models. In contrast in our numerical experiments the values of the specified variables at the lateral boundaries were generally free of this source of error.

However this technique of obtaining boundary data for the IBV problem also has a serious physical drawback. Atmospheric and oceanic motion comprise of an assemblage of interacting circulation systems with a wide spectrum of spatial and temporal scales. The a priori specification of the lateral boundary conditions severely disrupts the interaction between the larger scale motion and the limited area scale although the problem remains well posed mathematically. Ideally the two models with their differing spatial resolutions should be dynamically coupled.

On the other hand, global coverage of meteorological data for a given time is invariably acquired sequentially with the data for the local region being supplied first. Hence, there remains some advantage in using a "decoupled" boundary technique for short range limited area forecasts.

APPENDIX: On the Uniqueness of Solutions of the Shallow Water Equations

We first replace the height field h in system C with a new dependent variable c ($c^2 = gh$). We note that suitable authentic manipulation and rearrangement of the transformed versions of Eqs. (1c) yields the following slightly redundant set:

$$\frac{\partial}{\partial t}(u + 2c) + (u + c)\frac{\partial}{\partial x}(u + 2c) + v\frac{\partial}{\partial y}(u + 2c) + c\frac{\partial v}{\partial y} - fv = 0, \quad \text{(A1)}$$

$$\frac{\partial}{\partial t}(u - 2c) + (u - c)\frac{\partial}{\partial x}(u - 2c) + v\frac{\partial}{\partial y}(u - 2c) - c\frac{\partial v}{\partial y} - fv = 0, \quad \text{(A2)}$$

with similar equations for $(v + 2c)$ and $(v - 2c)$.

Proceeding as in [18], we can derive the following integral inequality,

$$\frac{\partial}{\partial t}\iint_S \{\mathbf{v}'^2 + 4c'^2\}\, dS \leqslant M \iint_S \{\mathbf{v}'^2 + 4c'^2\} + I \quad \text{(A3)}$$

with

$$I = -\oint_C \{\tfrac{1}{2}(\tilde{\mathbf{v}} \cdot \mathbf{n} - \tilde{c})(\mathbf{v}' \cdot \mathbf{n} - 2c')^2 + \tfrac{1}{2}(\tilde{\mathbf{v}} \cdot \mathbf{n} + \tilde{c})(\mathbf{v}' \cdot n + 2c')^2 + \tilde{\mathbf{v}} \cdot \mathbf{n}(\mathbf{v}' \cdot \mathbf{t})^2\}\, dc, \quad \text{(A4)}$$

where $\mathbf{v}' = \tilde{\mathbf{v}} - \mathbf{v}$ and $c' = \tilde{c} - c$ are the perturbation variables of the two solutions $(\mathbf{v}, c)$ and $(\tilde{\mathbf{v}}, \tilde{c})$, M is a positive constant with a value depending upon the magnitude of the spatial derivatives of both the velocity and geopotential fields, and $(\mathbf{t}, \mathbf{n})$ denote unit vectors in the tangential and outward normal directions to the curve c enclosing the area S.

It follows from Eqs. (A3) and (A4) that if the two flow fields are identical initially ($\mathbf{v}' = c' = 0$) and M remains finite, then the two flow fields remain indistinguishable if $\{\mathbf{v} \cdot \mathbf{n} - 2c\}$ is specified everywhere on C and the tangential velocity is specified at inflow points.

References

1. T. Nitta, *J. Meteor. Soc. Japan* **40** (1962), 13.
2. J. G. Charney, R. Fjortoft, and J. V. Neumann, *Tellus* **4** (1950), 237.
3. J. P. Gerrity and R. D. McPherson, *Monthly Weather Rev.* **97** (1969), 655.
4. H. H. Wang and P. Halpern, *J. Appl. Meteor.* **7** (1968), 525.
5. [illegible]
6. J. G. Charney, *Proc. Intern. Symp. Numerical Weather Prediction*, Tokyo (1962), 131.
7. M. A. Shapiro and J. O'Brien, *J. Appl. Meteor* **9** (1970), 345.
8. M. G. Wurtele, J. Paegle, and A. Sielecki, *Monthly Weather Rev.* **99** (1971), 537.
9. R. Asselin, *Atmosphere* **10** (1972), 44.
10. T. Matsuno, *J. Meteor. Soc. Japan* **44** (1966), 145.
11. R. D. Richtmyer and K. W. Morton, "Difference Methods for Initial-Value Problems," Interscience, New York, 1967.
12. H.-O. Kreiss and J. Oliger, Methods for the Approximate Solution of Time Dependent Problems, *GARP Publication Series No.* 10, Geneva, WMO 1973.
13. K. W. Morton, *Proc. Roy. Soc. London, Ser. A* **323** (1971), 237.
14. N. G. Campbell, *Numer. Math.* **14**, 435.
15. T. Elvius and A. Sundstrom, *Tellus* **25** (1973), 132.
16. A. Sundstrom, *Monthly Weather Rev.* **97** (1969), 340.
17. A. Sundstrom, Report DM-9 (1973), Institute of Meteorology, Stockholm.
18. H. C. Davies, *J. Atmospheric Sci.* **30** (1973), 147.
19. J. Smagorinsky, S. Manabe, and J. L. Holloway, *Monthly Weather Rev.* **93** (1965), 727.
20. A. Arakawa, in "Numerical Solution of Field Problems in Continuum Physics," *SIAM AMS Proc.* **2** (1970), 24.
21. N. A. Phillips, in "The Atmosphere and the Sea in Motion," p. 501, Rockefeller Institute Press, New York, 1959.
22. A. Arakawa, *J. Computational Phys.* **1** (1966), 119.
23. D. K. Lilly, *Monthly Weather Rev.* **93** (1965), 11.
24. B. Gustafsson, Comp. Sci. Report No. 33, Univ. of Uppsala, Uppsala.
25. B. Gustafsson, H.-O. Kreiss, and A. Sundstrom, *Math. Comp.* **26** (1970), 649.
26. N. A. Phillips, *Monthly Weather Rev.* **87** (1959), 333.